EXERCISE – III HINTS & SOLUTIONS

Sol.1 **(0, 0) ; (3, 27)**

$y = x^3$

Let $P(x_1, y_1)$ $\quad y_1 = x_1^3$

$\frac{dy}{dx} = 3x_1^2$

$3x_1^2 = y_1$

$3x_1^2 = x_1^3$

$x_1 = 0, 3$ $\quad$ Point (0, 0) (3, 27)

$y_1 = 0, 27$

Sol.2 **2x + y = 2**

$y = 1 + e^{-2x}$

$y = 2 \quad 2 = 1 + e^{-2x}$

$e^{-2x} = 1$

$x_1 = 0 \quad y_1 = 2$

Point (0, 2)

$\frac{dy}{dx} = -2e^{-2x}\Big|_p = -2$

$y - 2 = -2(x - 0)$

$y + 2x = 2$

Sol.3 **Tangent : x + y = 6, Normal x – y = 0**

$x^3 + y^3 = 6xy \quad (3, 3)$ $\quad$ Tangent

$3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$; $\quad y - 3 = -x + 3$

$y + x = 6 \quad (3, 3)$

$27 + 27\frac{dy}{dx} = 18 + 18\frac{dy}{dx}$; Normal

$\frac{dy}{dx} = -1 \quad y - 3 = 1(x - 3)$

$y = x$

Sol.4 **y = x**

$x^3 + y^3 = 8xy$

$y^2 = 4x$

solve for intersection point

$\frac{y^6}{64} + y^3 = 2y^3$

$y^3 + 64 = 128$

$y^3 = 64$

$y = 4 \quad x = 4$

point (4, 4)

$x^3 + y^3 = 8xy$

$3x^2 + 3y^2 y' = 8y + 8xy'$

at (4, 4)

$3(4)^2 + 3(4)^2 y' = 8(4) + 8(4)y'$

$3 + 3y' = 2 + 2y'$

$y' = -1$

Tangent $y - 4 = -1(x - 4) \Rightarrow y - 4 = -x + 4$

$y + x = 16$

Normal $y - 4 = x - 4 \Rightarrow y = x$

Sol.5 **(a) y – 2x – 3 = 0 $\quad$ (b) 2x + y – 7 = 0**

$y = x^2 - 2x + 7 \quad P(x_1, y_1)$

$\frac{dy}{dx} = 2x - 2\Big|_p = 2x_1 - 2$

(a) $2x_1 - 2 = 2$

$x_1 = 2 \quad y_1 = 4 - 4 + 7 = 7$

Tangent $y - 7 = 2(x - 2)$

$y - 7 = 2x - 4 \Rightarrow y = 2x + 3$

(b) $2x_1 - 2 = -2$

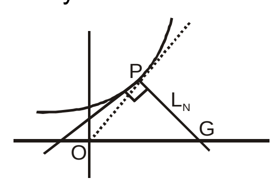
$x_1 = 0, \quad y_1 = 7$

$y - 7 = -2(x)$

$y + 2x = 7$

Sol.6

$x^2 - y^2 = a^2$



$P(x_1, y_1)$

we have to P.T.

$L_N = \sqrt{x_1^2 + y_1^2}$

$L_N = y_1\sqrt{1 + m^2}$ $\quad x^2 - y^2 = a^2$

$2x - 2yy' = 0$

$= y_1\sqrt{1 + \frac{x_1^2}{y_1^2}}$ $\quad y' = \frac{x}{y}$

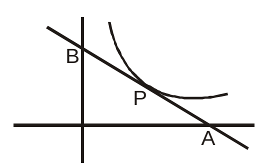
$= \sqrt{x_1^2 + y_1^2}$ H.P.

Sol.7 $x^{2/3} + y^{2/3} = a^{2/3}$

Let the point $P(a\cos^3\theta, a\sin^3\theta)$

$x = a\cos^3\theta, \quad y = a\sin^3\theta$

$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$



$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$\frac{dy}{dx} = -\tan\theta$

Tangent $y - a\sin^3\theta = -\tan\theta\,(x - a\cos^3\theta)$

$x\sin\theta + y\cos\theta = a\sin\theta\cos\theta$

$A(a\cos\theta, 0)$ $\quad B(0, a\sin\theta)$ $\quad \Rightarrow \quad AB = |a|$

Sol.8 $(4, 1), \left(-4, \frac{-31}{3}\right)$

$$\frac{dy}{dx} = \frac{1}{2}x^2, \ \frac{1}{2}x_1^2 = 8$$

$x_1 = \pm 4$

If $x_1 = +4$ and $x_1 = -4$

$y_1 = 11$ $\qquad y_1 = \frac{-31}{3}$

so point are (4, 1) and (–4, –31/3)

Sol.9 **(0, 0), (1, 2), (–1, –2)**

$y = 4x^3 - 2x^5$

Let the point $P(x_1, y_1)$

$y' = 4x_1^3 - 2x_1^5$ ....(1)

$$\frac{dy}{dx} = 12x^2 - 10x^4\Big|_p = x_1^2(12 - 10x_1^2)$$

Tangent

$y - y_1 = M_T\,(x - x_1)$

$y - y_1 = x_1^2\,(12 - 10x_1^2)\,(x - x_1)$

passes through origin

$-y_1 = x_1^2\,(12 - 10x_1^2)\,(0 - x_1)$

$y_1 = x_1^3\,(12 - 10x_1^2)$

$4x_1^3 - 2x_1^5 = 12x_1^3 - 10x_1^5$

$4 - 2x_1^2 = 12 - 10x_1^2$

$x_1 = 0$ $\qquad x_1^2 = 1$

$y_1 = 0$ $\qquad x_1 = \pm 1$

$x_1 = 1 \Rightarrow y_1 = 2$

$x_1 = -1 \Rightarrow y_1 = -2$

Points (0, 0) (1, 2) (–1, –2)

Sol.10 **(9/4, 3/8)**

$y^2 = x(2 - x)^2$

$2yy' = (2 - x)^2 - 2x(2 - x)$

at (1, 1)

$2y' = 1 - 2 \Rightarrow y' = -1/2$

Tangent $y - 1 = -\frac{1}{2}\,(x - 1)$

$2y - 2 = -x + 1$

$2y + x = 3$

Intersection of the tangent and curve is given by

$$\frac{1}{4}(-x_1 + 3)^2 = x_1(4 + x_1^2 - 4x_1)$$

$(x_1 - 1)\,(4x_1^2 - 13x_1 + 9) = 0$

$(x_1 - 1)^2\,(4x_1 - 9) = 0$

$x_1 = 1$ is already the point of tangency

$x_1 = 9/4 \Rightarrow y_1 = 3/8$

Point (9/4, 3/8)

Sol.11 $\boldsymbol{\frac{\pi}{3}}$

$y = 2\sin^2 x$ $\qquad y = \cos 2x$

For intersection point

$2\sin^2 x = 1 - 2\sin^2 x$

$\sin x = \pm\frac{1}{2}$

$y = 2\sin^2 x$ $\qquad y = \cos 2x$

$m_1 = 2\sin 2x$ $\qquad m_2 = -2\sin 2x$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{4\sin 2x}{1 - 4\sin^2 2x}\right| = \left|\frac{8\left(\pm\frac{1}{2}\right)\left(\pm\frac{\sqrt{3}}{2}\right)}{1 - 16\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)}\right|$$

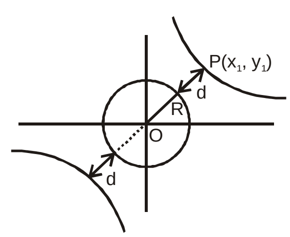
$\tan\theta = \sqrt{3}$ $\qquad \Rightarrow \theta = \frac{\pi}{3}$

Sol.12 $\boldsymbol{3\sqrt{2} - 1}$

$xy = 9 \Rightarrow x_1y_1 = 9$ $\qquad x^2 + y^2 = 1$

$xy' + y = 0$

$y' = -\frac{y}{x}$



$M_T = -\frac{x_1}{y_1}$

$M_N = \frac{x_1}{y_1} = \frac{y_1}{x_1}$

$x_1^2 = y_1^2$ $\qquad y_1 = \frac{9}{x_1}$

$x_1 = \pm 3$

Point (3, 3) $\qquad$ (–3, –3)

P(3, 3)

$OP = \sqrt{9 + 9} = 3\sqrt{2}$

OR = 1 (Radius)

Shortest Distance $d = RP = 3\sqrt{2} - 1$

Sol.13 (–6, 3)

$3x^2 - 4y^2 = 72$

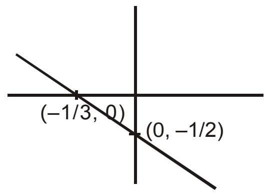
$3x + 2y + 1 = 0$

Let the point $P(x_1, y_1)$

$3x_1^2 - 4y_1^2 = 72$ ....(1)

$6x - 8yy' = 0$

$$y' = \left.\frac{3x}{4y}\right|_p = \frac{3x_1}{4y_1}$$



$$\frac{3x_1}{4y_1} = -\frac{3}{2}$$

$\Rightarrow \quad x_1 = -2y_1$ ....(2)

from (1) and (2) Points (3, –6)

$y_1 = \pm 3, \quad x_1 = \mp 6$ (–3, 6)

$3x + 2y + 1 = 0$

(3, –6) (–3, 6)

$$d_1 = \left|\frac{9-12+1}{\sqrt{9+4}}\right| \qquad d_2 = \left|\frac{-9+12+1}{\sqrt{9+4}}\right|$$

$$= \left|\frac{2}{\sqrt{3}}\right| \qquad d_2 = \left|\frac{4}{\sqrt{3}}\right|$$

$d_2 > d_1$

so point (3, –6) is the point from distance is min.

Sol.14 $x^2y^2 = a^2(x^2 - a^2)$

$2xy^2 + 2x^2yy' = 2a^2x$

$xy^2 + x^2yy' = a^2x$

$$y' = \frac{a^2x - xy^2}{x^2y}$$

$$y' = \frac{a^2x^2 - x^2y^2}{x^2y^2} \times \left.\left(\frac{y}{x}\right)\right|_p$$

$$= \left.\frac{a^4}{x^3y}\right|_p$$

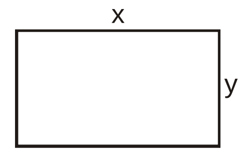
$$y' = \frac{a^4}{x_1^3 y_1} = m$$

$L_{SN} = y_1 m$

$$L_{SN} = \frac{a^4}{x_1^3} \qquad \text{H.P.}$$

Sol.15 $\frac{dx}{dt} = -3$ cm/min

$\frac{dy}{dt} = 2$ cm /min



(i) $P = 2x + 2y$

$$\frac{dp}{dt} = 2\frac{dx}{dt} + 2\frac{dy}{dt}$$

$= -6 + 4 = -2$ cm/min.

(ii) $A = xy$

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$= 10(2) + 6(-3) = 2$ cm²/min.

Sol.16 zero

$y = x - x^2$ at $x = 1, y = 0$

Area of second square $A_2 = y^2$

$$\frac{dA_2}{dt} = 2y\frac{dy}{dt}$$

$$\frac{dA_1}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dA_2}{dA_1} = \frac{y}{x}\,\frac{dy}{dx}$$

$$\frac{dA_2}{dA_1} = 0$$

Sol.17 –1

$$I = \int_a^b f'(x).f''(x)\,dx$$

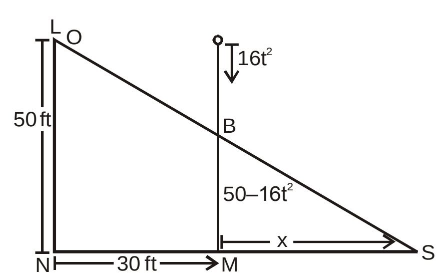
$$= \left.\frac{[f'(x)]^2}{2}\right|_a^b = \frac{1}{2}\,[f'(b) - f'(a)]\,[f'(b) + f'(a)]$$

$\tan\theta = f'(b) \Rightarrow f'(b) = 1$

$f'(a) = \tan 60^\circ = \sqrt{3}$

$$I = \frac{1}{2}(1 - \sqrt{3})(1 + \sqrt{3}) = -1$$

Sol.18 –1500 ft/sec.



By triangle property

$$\frac{x+30}{50} = \frac{x}{50-16t^2}$$

$50x - 16x\,t^2 + 1500 - 480\,t^2 = 50x$

$-16xt^2 + 1500 - 480\,t^2 = 0$

$$-16\frac{dx}{dt}\cdot t^2 = 32x\cdot t + 960\,t$$

$$-16 \times \frac{1}{4} \times \frac{dx}{dt} = 32\,(345)\left(\frac{1}{2}\right) + 960 \times \frac{1}{2}$$

$$\frac{dx}{dt} = -1500 \text{ ft/sec.}$$

Sol.19 $\pm \frac{c}{\sqrt{2}}$

$xy = (c-x)^2$

Diff w.r.t. x $\qquad x_1y_1 = (c - x_1)^2$

$xy' + y = -2(c - x)$

$$y' = \frac{-2(c-x)-y}{x}\Bigg|_p = \frac{-2(c-x_1)-y_1}{x_1}$$

$$= \frac{-2(c-x_1)}{x_1} - \frac{(c-x_1)^2}{x_1^2} = -1$$

$$\left(\frac{c-x_1}{x_1}\right)\left(2+\frac{c-x_1}{x_1}\right) = 1$$

$$\frac{c^2-x_1^2}{x_1^2} = 1$$

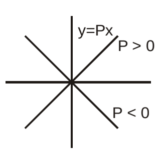
$c^2 - x_1^2 = x_1^2$

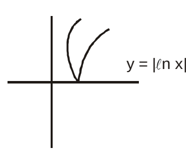
$$x_1 = \pm\frac{c}{\sqrt{2}}$$

Sol.20 $p \in (0, 1/e)$

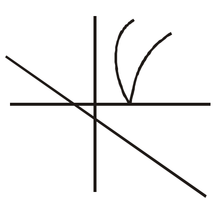
$y = px$

$y = |\ell n\, x|$



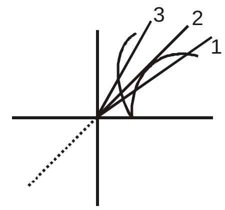


case-I If $p < 0$



No solution

case-II If $p > 0$



For second posibility

$y = px \quad y = \ell n\, x$

$$\frac{dy}{dx} = p \quad \frac{dy}{dx} = \frac{1}{x}$$

$$p = \frac{1}{x_1} \Rightarrow px_1 = 1 \ ...(1)$$

Three posibility are there $y_1 = px_1$ & $y_1 = \ell n\, x_1$

$y_1 = 1 \qquad I = \ell n\, x_1$

$x_1 = e$

$y_1 = px_1$

$1 = pe$

$0 < p < \frac{1}{e} \qquad p = 1/e$

Sol.21 **8**

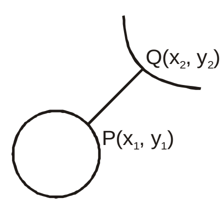
$$(x_1 - x_2)^2 + \left(\sqrt{2-x_1^2} - \frac{9}{x_2}\right)^2 \quad x_1 \in (0, \sqrt{2}\,)$$

Its similar to distance formula. $\qquad x_2 \in R^+$

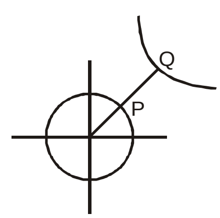
$(x_1 - x_2)^2 + (y_1 - y_2)^2$

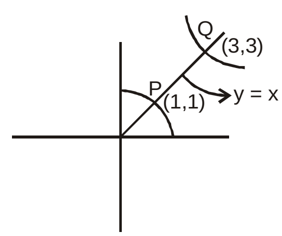
$y_1 = \sqrt{2-x_1^2}$

$y_1^{\ 2} + x_1^{\ 2} = 2$



$$y_2 = \frac{9}{x_2} \Rightarrow x_2y_2 = 9$$





$(x_1, y_1) = (1, 1)$

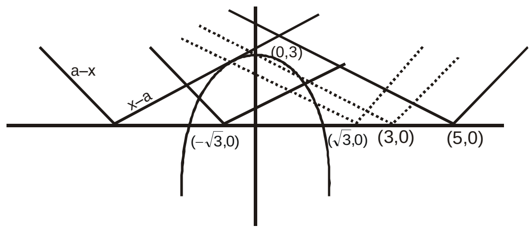
$(x_2, y_2) = (3, 2)$

$$(x_1 - x_2)^2 + \left(\sqrt{2-x_1^2} - \frac{9}{x_2}\right)^2 = 8$$

Sol.22 $a \in \left(-\frac{13}{4}, 3\right)$

$3 - x^2 > |x - a|$ If a = 5

$|x-5|$ → $x - 5 \quad x \ge 5$; $5 - x \quad x < 5$



$y = 3 - x^2$

$\frac{dy}{dx} = -2x = 1$

$x_1 = -\frac{1}{2}$

$y_1 = 3 - \frac{1}{4} = \frac{11}{4}$

$y = x - a$

$\frac{11}{4} = -\frac{1}{2} - a \quad \Rightarrow \quad a = -\frac{13}{4}$

$a \in \left(-\frac{13}{4}, 3\right)$

Sol.23 $\frac{\mathbf{8b}}{\mathbf{27}}$

$by^2 = (x + a)^3$

$2by\, y' = 3(x + a)^2$

$m = y' = \left.\frac{3(x+a)^2}{2by}\right|_p = \frac{3(x_1+a)^2}{2by_1}$

P point will lie on the curve also

$by_1^2 = (x_1 + a)^3$

$L_{ST} = \frac{y_1}{m}$ $\qquad L_{SN} = y_1 m$

$pL_{SN} = \text{or } (L_{ST})^2$

$\frac{p}{q} L_{SN} = (L_{ST})^2$

$\frac{p}{q}(y_1 m) = \frac{y_1^2}{m^2}$

$\frac{p}{q} = \frac{y_1}{m^3} = \frac{y_1 \times 8b^3 y_1^3}{27(x_1+a)^6}$

$\frac{p}{q} = \frac{8b}{27}$

Sol.24 $x = a(\cos t + \log \tan \frac{t}{2})$

$\frac{dx}{dt} = a(-\sin t + \frac{1}{\tan\frac{t}{2}} \sec^2 \frac{t}{2} \cdot \frac{1}{2})$

$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\sin t}\right)$

$y = a \sin t$

$\frac{dy}{dt} = a \cos t$

$\frac{dy}{dx} = \frac{a\cos t \; \sin t}{a(1-\sin^2 t)} = \tan t$

$y - a \sin t = \tan t\,(x - a \cos t - a \log \tan \frac{t}{2})$

at x-axis, y = 0

$-a \cos t = x - a \cos t - a \log \tan \frac{t}{2}$

$x = a \log \tan \frac{t}{2}$

point $Q\left(a \log \tan \frac{t}{2}, 0\right)$

$P\left(a(\cos t + \log \tan \frac{t}{2}), a \sin t\right)$

$PQ = \sqrt{a^2\cos^2 t + a^2 \sin^2 t} = |a|$

Sol.25 $6y = x^3 + 2$

$6\frac{dy}{dx} = 3x^2 \frac{dx}{dt}$

Given that $\frac{dy}{dt} = 8\frac{dx}{dt}$

$6 \times 8 = 3x^2$

$x^2 = 16$

$x = \pm 4$

$x = 4 \Rightarrow y = 11$

$x = -4 \Rightarrow y = -31/3$

(4, 11) & (–4, –31/3)